Indian Statistical Institute, Bangalore Centre
M.Math. II Year, I Semester 2008-09

Subject: Number Theory **Mid-Sem Examination**
Max. Marks: 100 **Date: 22/09/2008** **Duration 3 hours**

1. Let p be an odd prime.
(a) Show that -1 is a square modulo p if and only if $p \equiv 1 (\text{mod } 4)$.
(b) Show that 2 is a square modulo p if and only if $p \equiv \pm 1 (\text{mod } 8)$.
(c) Use the law of quadratic reciprocity to compute the following Legendre symbols: [5+10+10=25]

$$\left(\frac{17}{29}\right), \left(\frac{59}{263}\right), \left(\frac{111}{397}\right), \left(\frac{210}{397}\right).$$

2. For $n \geq 1$, let F_n denote the Fermat Number $2^{2^n} + 1$.
(a) Show that for $m \neq n$, F_m and F_n are relatively prime.
(b) The $\pi(x)$ denote the number of primes $\leq x$. Use part (a) to prove that there is a constant $c > 0$ such that $\pi(x) \geq c \log \log x$ for all $x \geq 2$.
(c) If p is a prime dividing F_n then show that $p \equiv 1 (\text{mod } 2^{n+1})$.

[8+8+9=25]

3. Let G be the set of all multiplicative functions $f : \mathbb{N} \to \mathbb{C}$ which are not identically zero.
(a) Show that G, equipped with Dirichlet convolution $*$, is an abelian group.
(b) Show that G has no non-identity element of finite order. [12+13=25]

4. (a) State and prove Euler's product formula for the Riemann Zeta function.
(b) Use part (a) to prove that the sum of reciprocals of all the primes is infinity. [15+10=25]